Direction cosines and Projections

Article 1. Direction cosines of a line.

If α,β,γ be the angles that a given line makes with positive directions of co-ordinates axes , then $\cos\alpha,\cos\beta,\cos\gamma$ are called the direction cosines of the given lines, and are generally denoted by < l,m,n >. Thus $l=\cos\alpha, m=\cos\beta, n=\cos\gamma$.

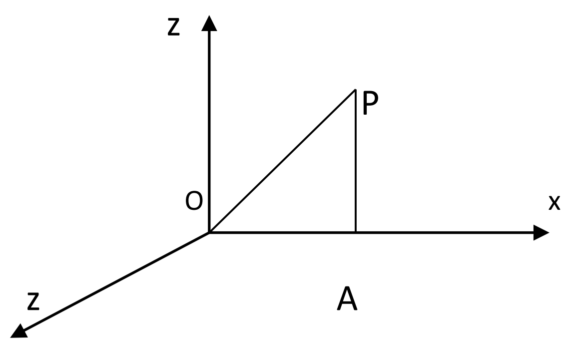
Cor. The Direction cosines of the axes of co-ordinates.

The x-axis makes angle $0^\circ, 90^\circ, 90^\circ$ with the coordinates axes , its Direction cosines are $<\cos 0^\circ, \cos 90^\circ, \cos 90^\circ>$ i.e. <1,0,0 > .

similarly , the Direction cosines of y-axis are < 0,1,0 > and Direction cosines of z-axis are < 0,0,1 > .

Article 2. If l,m,n be the d.c's of a line OP and OP = r , then the co-ordinates of P are (lr,mr,nr)

Let (x,y,z) be the co-ordinates of P , and A , the foot of perpendicular from P on x-axis . Then x = OA



If OP = r then

$$\cos\alpha = \frac{OA}{OP}, \Rightarrow l = \frac{x}{r}$$

x = lr, similarly y= mr and z = nr

Thus the co-ordinates of P are (lr,mr,nr)

Cor. Squaring and adding , $x^2+y^2+z^2=r^2\left(l^2+m^2+n^2\right)$

$$\Rightarrow r^2 = r^2\left(l^2+m^2+n^2\right)$$

Hence $1=l^2+m^2+n^2$.

Example 1. What are the d.c's of line equally inclined in the axes ?

Sol . If a line makes acute angles α,β,γ with the axes we have $\alpha=\beta=\gamma$

$\therefore \cos\alpha=\cos\beta=\cos\gamma$ or l= m =n

Since $1=l^2+m^2+n^2$, $\therefore l^2+l^2+l^2=1$

Or $3l^2=1$

Hence the d.c's of a line are $<\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}>$.

Article 3. Direction ratios of a line

Def. Any three numbers , which are proportional to the d.c's of a line , are called the d.r's of a line.

To find the actual d.c's of a line whose proportional d.c's (i.e. d,r's) are given.

Let a,b,c be the d.r's of a line and let l,m,n be the actual d.c's of the line.

Then $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}=\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}}$

$$\Rightarrow \frac{l}{a}=\frac{m}{b}=\frac{n}{c}=\frac{1}{\sqrt{a^2+b^2+c^2}}$$

$$\therefore l=\frac{a}{\sqrt{a^2+b^2+c^2}},m=\frac{b}{\sqrt{a^2+b^2+c^2}},n=\frac{c}{\sqrt{a^2+b^2+c^2}}$$

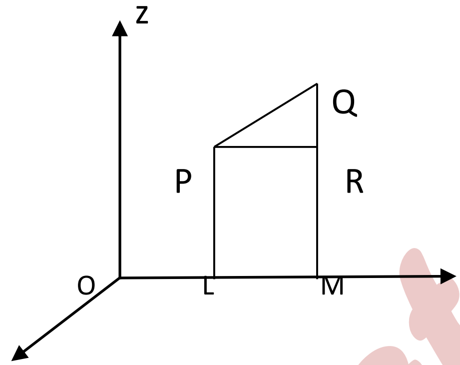
Article 4.Direction cosines of a line joining two points.

. Let P(x_1,y_1,z_1) and Q(x_2,y_2,z_2) be the coordinates of any two points and l,m,n be the d.c's of the line PQ. Then if α,β,γ be the angles which the line PQ makes with the axes , $l=\cos\alpha,m=\cos\beta,n=\cos\gamma$

Draw PL,QM perpendicular on x-axis . Draw PR $\perp$ QM,

so that PR = LM = OM- OL

$= x_2 - x_1$.



Now from triangle PQR ,

$$\cos\alpha = \frac{PR}{PQ} = \frac{x_2 - x_1}{PQ}.$$

Similarly $\cos\beta = \frac{y_2 - y_1}{PQ}$, $\cos\gamma = \frac{z_2 - z_1}{PQ}$

Hence the d.c's of PQ are $< \frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ} >$

Or the d.r's are $< x_2 - x_1, y_2 - y_1, z_2 - z_1 >$.

Article 5. To find the angle between two lines whose d.c's are $< l_1, m_1, n_1 >$ and $< l_2, m_2, n_2 >$ respectively.

Through the origin O draw two lines OP and OQ of length r and s respectively.

Let θ be the angle between these lines.

Then co-ordinates of $< l_1 r, m_1 r, n_1 r >$ and $< l_2 s, m_2 s, n_2 s >$ respectively.

$$PQ^2 = (l_2 s - l_1 r)^2 + (m_2 s - m_1 r)^2 + (n_2 s - n_1 r)^2$$

$$= s^2(l_2^2 + m_2^2 + n_2^2) + r^2(l_1^2 + m_1^2 + n_1^2) - 2rs(l_1 l_2 + m_1 m_2 + n_1 n_2)$$

$$= s^2 + r^2 - 2rs(l_1 l_2 + m_1 m_2 + n_1 n_2).$$

Now in triangle OPQ , by cosine formula,

$$\cos\theta = \frac{OP^2 + OQ^2 - PQ^2}{2OP.OQ}$$

$$= \frac{r^2 + s^2 - [s^2 + r^2 - 2rs(l_1l_2 + m_1m_2 + n_1n_2)]}{2rs}$$

$$= l_1l_2 + m_1m_2 + n_1n_2$$

Cor 1. Condtion of perpendicularity

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

Cor 2. Condition of Parallelism.

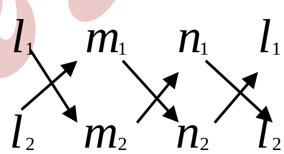
$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}.$$

Cor 3. Expressions for $\sin\theta$ and $\tan\theta$.

$$\sin\theta = \sqrt{\sum(l_1m_2 - m_1l_2)^2}$$

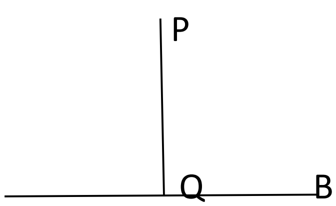
method to write down the result

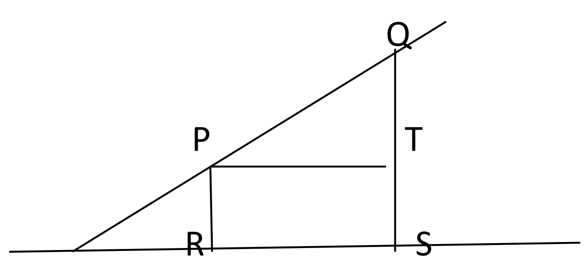
$$\tan\theta = \frac{\sqrt{\sum(l_1m_2 - m_1l_2)^2}}{\sum l_1l_2}$$



Article 6. Projection on a line.

(a) The Projection of a point on a line is the foot of perpendicular from the point on the line. Thus Projection of P on AB is Q.

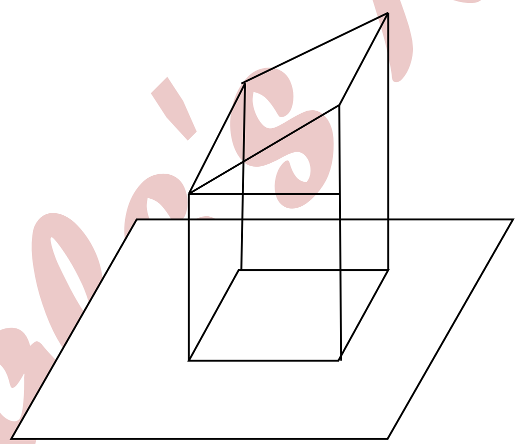




(b) The Projection of a segment of a line is the line joining the feet of perpendicular from its ends on the line. Thus RS is the Projection of PQ on the line AB and length of Projection is RS = PQ cos θ .

(c) The Projection of the join of two points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) On a line whose d.c's are l,m,n is $l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$.

(d) The projection of plane area bounded by a curve is the area enclosed by the curve through the feet of perpendiculars from different points on the original curve to the plane.
Thus the projection of area of rectangle ABCD on the plane is the area PQRS.



Area of projection PQRS= PO.PS= AB cosθ .AD = AB.AD cosθ

= ABCD cosθ = rectangle ABCD . cosθ .

The Plane

Definition of a Plane. A Plane is a surface such that if any two points on it are taken , then the line joining them lies wholly in the plane.

From the above def. of a plane we shall prove our result. Let P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) be the coordinates of any two points lying on the locus whose equation is

ax + by + cz + d = 0 .... (1)

$$\therefore ax_1 + by_1 + cz_1 + d = 0 \qquad \text{.....................} \quad (2)$$

and $$ax_2 + by_2 + cz_2 + d = 0 \qquad \text{......................} \quad (3)$$

Multiplying (3) by m_1 and (2) by m_2 and adding we get

$$a(m_2 x_1 + m_1 x_2) + b(m_2 y_1 + m_1 y_2) + c(m_2 z_1 + m_1 z_2) + d(m_2 + m_1) = 0.$$

Or $$a\left(\frac{m_2 x_1 + m_1 x_2}{m_1 + m_2}\right) + b\left(\frac{m_2 y_1 + m_1 y_2}{m_1 + m_2}\right) + c\left(\frac{m_2 z_1 + m_1 z_2}{m_1 + m_2}\right) + d = 0$$

Which clearly shows that the point $\left(\frac{m_2 x_1 + m_1 x_2}{m_1 + m_2}, \frac{m_2 y_1 + m_1 y_2}{m_1 + m_2}, \frac{m_2 z_1 + m_1 z_2}{m_1 + m_2}\right)$ lies on (1)

Thus, we have shown that the line joining P and Q completely lies on (1) which is possible only when the surface (1) is a plane.

Thus, a general equation of first degree in x,y,z always represents a plane.

Cor . (One point form)

Let the equation of a plane be ax +by +cz + d=0 .... (1)

Since it passes through (x_1, y_1, z_1) $\therefore ax_1 + by_1 + cz_1 + d = 0$ ....(2).

Subtracting (2) from (1) , we have $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

Note.The general equation of a plane be ax +by +cz + d=0 or $\frac{a}{d}x + \frac{b}{d}y + \frac{c}{d}z + 1 = 0$.

The above form shows that there are three arbitrary constants occurring in the equation of the plane. Hence a plane can be found to satisfy three conditions, each condition giving us the value of a constant.

Normal form of the equation of a plane.

Art 1.To find the equation of a plane in terms of p ,the length of the perpendicular from the origin on the plane , and l,m,n the d.c's of this perpendicular.

Let ABC be plane and ON be the perpendicular from O on the plane , so that ON=p and d.c's of ON are l,m,n .

Let P(x,y,z) be any point on the plane . Join OP and PN . Then $\angle ONP = 90^\circ$.

We know that if a line is $\perp$ to a plane , then it is $\perp$ to every st. line in the plane which meets it in that point.

$\therefore$ ON = projection of OP on ON

$\Rightarrow p$ = l(x-0) +m (y- 0) + n (z-0)

Or p = lx +my +nz

Note. (1) p is always positive

(2) we find that d.c's of the normal to the plane represented by ax +by +cz +d =0 are proportional to a ,b and c. in other words, the d.c's are proportional to the coefficients of x , y and z.

Example 1. Reduce the equation of plane 6x -3y -2z +5=0 in normal form.

Sol. Equation of plane is 6x -3y -2z +5 =0

$\Rightarrow$ 6x -3y -2z = -5

$\Rightarrow$ -6x + 3y +2z =5

Dividing by $\sqrt{36+9+4}=7$

$$\Rightarrow \frac{-6}{7}x+\frac{3}{7}y+\frac{2}{7}z=\frac{5}{7}$$

Example 2. O is the origin and P is the point (1,2,3).Find the equation of a plane through P at right angle to OP.

Sol. If P is the point (1,2,3) , then the plane passes through P and is at right angle to OP . So d.r's of the line OP is < 1-0, 2-0,3-0 > i.e. < 1,2,3>

Hence equation of plane is 1(x-1) + 2 (y -2) + 3 (z-3) = 0 (one point form)

$\Rightarrow$ X + 2y +3z – 14 =0.

Example 3. If the line drawn from P (1,0,2) meets a plane right angles at the point Q(2,-1,3) , then find the equation of the plane.

Sol. The plane passes through Q (2,-1,3) and is at right angle to PQ . so d.r's of the line PQ is < 2-1, -1-0, 3-2 > i.e. < 1 ,-1,1 >

Hence equation of plane is 1(x-2) -1 (y +1) + 1 (z-1) = 0 (one point form)

$\Rightarrow$ x –y + z - 4 =0.

Example 4. Find the equation of the plane through the three points (0,1,1) , (1,1,2) ,(-1,2,-2).

Sol. Let equation of plane passing through (0,1,1) is

a(x-0) + b(y – 1) + c (z-1) =0 ……………… (1) (one point form)

since it also passes through (1,1,2) and (-1,2,-2)

$\therefore$ a + 0b + c =0 …………………… (2)

and -a + b -3c =0 ……………………………(3)

Solving (2) and (3) , we have

$$\frac{a}{0-1}=\frac{b}{-1+3}=\frac{c}{1-0}$$
$$or \quad \frac{a}{-1}=\frac{b}{2}=\frac{c}{1}$$

Putting these values of a,b,c in (1)

-1 (x-0) +2 (y-1) +1 (z-1) =0

Or -x +2y + z -3=0

Which is required equation of plane.

Example 5. Show that the points (0,-1,0) ,(2,1,-1) , (1,1,1) , (3,3,0) are coplanar.

Sol. Let us first find the equation of the plane through three points (0,-1,0) , (2,1,-1) , (1,1,1) .

Let equation of plane passing through (0,-1,0) is

a(x-0) + b(y +1) + c (z-0) =0 ……………… (1) (one point form)

since it also passes through (2,1,-1) and (1,1,1)

$\therefore$ 2a + 2b - c =0 …………………… (2)

and a + 2b +c =0 ……………………………(3)

Solving (2) and (3) , we have

$$\frac{a}{2+2}=\frac{b}{-1-2}=\frac{c}{4-2}$$
$$or \quad \frac{a}{4}=\frac{b}{-3}=\frac{c}{2}$$

Putting these values of a,b,c in (1)

4x – 3(y + 1) +2 z=0

Or 4x -3y +2z -3 =0

The four given points will be coplanar if the fourth remaining point (3,3,0) lies on the plane

If 4 (3) – 3 (3) + 2 (0) – 3 =0

If 12- 9 + 0 – 3 =0

If 0 = 0 which is true.

Example 6.Prove that the line joining the points P(2,2,-1) , Q(3,4,2) intersect the line joining the points R(7,0,6) , S(2,5,1).

Sol. The four given points are P(2,2,-1) , Q(3,4,2), R(7,0,6) and S(2,5,1).

We are to prove that the line joining P,Q intersects the line joining R, S. The line PQ intersect RS , if P,Q, R ,S lie on the same plane.Thus we are to prove that all the four given points are coplanar.

Proceed as in example 5.

Example 7.If from the point P (a,b,c) , perpendiculars PQ , PR be drawn to XY and YZ planes ,find the equation of plane OQR.

Sol. The co-ordinates of Q , the foot of perpendicular from P (a,b,c) on XY plane is (a,b,0) .

The co-ordinates of R , the foot of perpendicular from P (a,b,c) on YZ plane is (0,b,c).

Let equation of plane passing through (0,0,0) is

A(x-0) + B(y -0) + C (z-0) =0 ……………… (1) (one point form)

since it also passes through (a,b,0) and (0,b,c)

∴ aA + bB +0C =0 …………………… (2)

and 0A + bB +cC =0 ……………………………(3)

Solving (2) and (3) , we have

$$\frac{A}{bc-0}=\frac{B}{0-ac}=\frac{C}{ab-0}$$
$$or \quad \frac{A}{bc}=\frac{B}{-ac}=\frac{C}{ab}$$

Putting these values of A,B,C in (1)

bc (x-0) – ac y +ab z =0

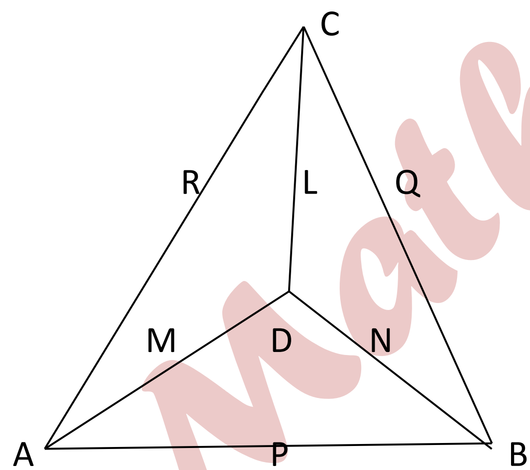
or bcx –acy +abz =0

or $\frac{x}{a}-\frac{y}{b}+\frac{z}{c}=0$

Example 8. Show that the six planes each passing through one edge of a tetrahedron and bisecting the opposite edge meet in a point.

Sol. Let $A(x_1,y_1,z_1), B(x_2,y_2,z_2), C(x_3,y_3,z_3), D(x_4,y_4,z_4)$ be the vertices of the tetrahedron ABCD.

Let L,M,N,P,Q,R be the mid points of the various edges as shown.



Now consider the plane ABL, through the AB and passing through L (i.e. bisecting the opposite edge CD)

L is $\left(\frac{x_3+x_4}{2},\frac{y_3+y_4}{2},\frac{z_3+z_4}{2}\right)$.

Since line AB lies on this plane ,

$$\therefore P\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$

The mid point of AB also lies on this plane . Again since P and L lie in the plane , so the line PL wholly lies in this plane and so the mid point of PL

$$G\left(\frac{x_1+x_2+x_3+x_4}{4},\frac{y_1+y_2+y_3+y_4}{4},\frac{z_1+z_2+z_3+z_4}{4}\right)$$ is

also lies on this plane.

Continuing the same argument we can show that G lies on other five planes also which pass through the other edges and bisect the opposite edges.

Example 9.Find the equation of the plane through the point (1,-2,3) and perpendicular to the planes x – y + 2z =3 and 3 x + 2y – z =4.

Sol. Let equation of plane passing through (1,-2,3) is a(x-1) + b(y +2) + c (z-3) =0 ……………… (1) (one point form)

plane (1) is perpendicular to the planes x – y + 2z =3 and 3 x + 2y – z =4.

$\therefore$ a – b + 2c =0 ……………….(2)

and 3a +2b –c =0 ………………..(3)

solving (2) and (3) ,we get

$$\frac{a}{1-4}=\frac{b}{6+1}=\frac{c}{2+3} \quad or \quad \frac{a}{-3}=\frac{b}{7}=\frac{c}{5}.$$

Put in (1) -3(x-1) + 7(y +2) + 5 (z-3) =0

-3x + 7y + 5z +2 =0.

Example 10. Example 9.Find the equation of the plane through the points (2,1,-1) and (-1,3,4) and perpendicular to the plane x – 2y + 4z =10 .

Sol. Let equation of plane passing through (2,1,-1) is

a(x-2) + b(y -1) + c (z+1) =0 ……………… (1) (one point form)

since it also passes through (-1,3,4)

$\therefore$ -3a + 2b +5c =0 …………………… (2)

Also plane (1) is perpendicular to x-2y+4z=10.

$\therefore$ a - 2b +4c =0 …………………………….(3)

Solving (2) and (3) , we get $\frac{a}{8+10}=\frac{b}{5+12}=\frac{c}{6-2} \quad or \quad \frac{a}{18}=\frac{b}{17}=\frac{c}{4}$

Put in (1) 18(x-2) + 17(y -1) + 4(z+1) =0

$\Rightarrow$ 18x +17y +4z =49.

EXERCISE 1 (a)

1.Find the equation of a plane through the points (3,-1,2) , (1,-1,-3) and (4,-3,1).

2. Show that the four points (0,-1,-1) , (4,5,1) ,(3,9,4) and (-4,4,4) lie on a plane.

3. Show that P(-1,4,-3) is the circumcentre of the triangle formed by the points A(3,2,-5),B (-3,8,-5) and C(-3,2,1).

(Hint ; (a) PA=PB=PC (b) P, A , B , C are coplanar.)

4.Find the equation of the plane through the points (1,-1,2) and (2,-2,2) and perpendicular to the plane 6 x – 2y + 2z =9.

5. Find the equation of the plane through the point (-1,3,2) and perpendicular to the planes x +2 y + 3z =5 and 3 x + 3y + z =0.

6.Find the equation of the plane that passes through (2,-3,1) and is perpendicular

To the line joining the points (3,4,-1) and (2,-1,5).

ANSWERS

1.10x + 7y – 4z -15 =0.

4. x+y -2z +4=0

5.7x – 8y + 3z +25=0

6. x+5y-6z+19=0.

Article 2.(Intercepts form) . To find the equation to the plane in terms of the intercepts a,b and c which it makes on the axes.

Let the equation of the plane be Ax + By + Cz + D =0 ....(1)

Since it makes intercepts a, b ,c on the axes of x ,y ,z respectively, then (a,0,0), (0,b,0) , (0,0,c) will satisfy (1).

$\therefore$ Aa + D = 0 , Bb + D = 0, Cc + D = 0.

$$\therefore A = \frac{-D}{a}, B = \frac{-D}{b}, C = \frac{-D}{c}$$

Putting the values of A, B , C in (1) , we get

$$\frac{-D}{a}x - \frac{-D}{b}y - \frac{-D}{c}z + D = 0$$
$$or \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Is the required equation of the plane.

Article 3.Perpendicular distance of a point from a plane.

To find the perpendicular distance of a given point from a given plane.

Let us first consider the equation of a plane ABC in the normal form

lx +my +nz =p. ....(1)

Let $P(x_1, y_1, z_1)$ be the point from which the perpendicular distance of the plane ABC is to be calculated. Draw PM perpendicular from P to the plane (1) . Also draw a plane parallel to (1) passing through P. Its equation is given by

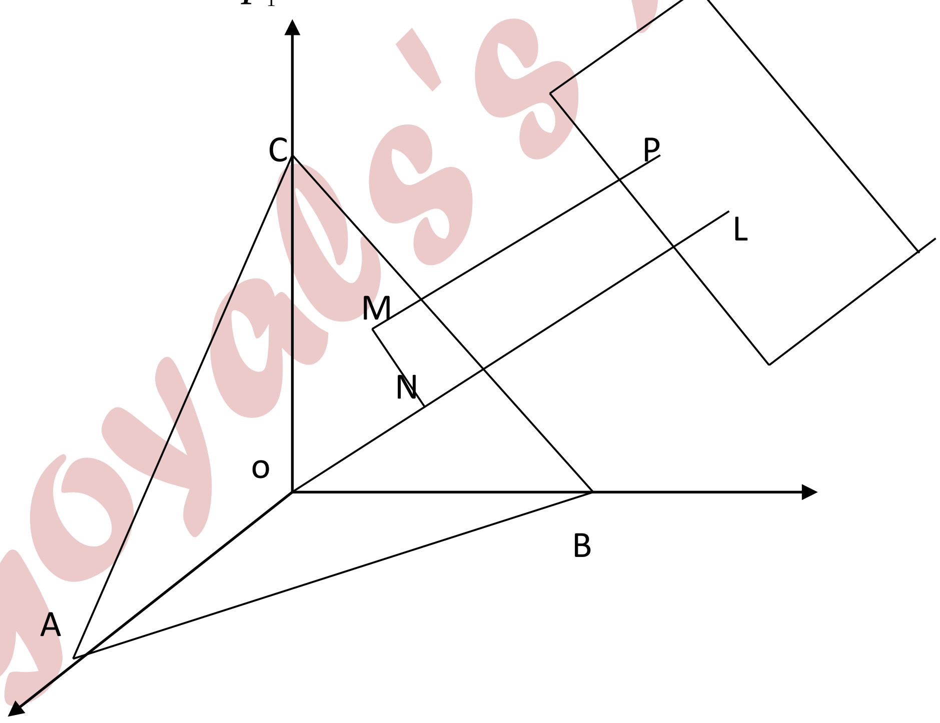
lx + my +nz = p_1 . ...(2)

Where p_1 is the length of the normal drawn from the origin to the plane (2) . But the plane (2) passes through P . Therefore

$lx_1 + my_1 + nz_1 = p_1$ . ...(3)

Now draw OL perpendicular to the plane (2) which meets the plane (1) in N. From fig we have OL = p_1 and ON = p.

Thus MP = NL = OL-ON = p_1 -p.



Hence the perpendicular distance of the point P from the plane is lx +my +nz =p

Is $lx_1 + my_1 + nz_1 - p$.

Consider nowthe general equation of the plane ax +by +cz +d =0 ....(4)

This can be written in the normal forms as

$$\frac{a}{\sqrt{a^2+b^2+c^2}}x+\frac{b}{\sqrt{a^2+b^2+c^2}}y+\frac{c}{\sqrt{a^2+b^2+c^2}}z+\frac{d}{\sqrt{a^2+b^2+c^2}}=0$$

Hence the perpendicular distance of the point P from the plane (4) is

$$\frac{a}{\sqrt{a^2+b^2+c^2}}x_1+\frac{b}{\sqrt{a^2+b^2+c^2}}y_1+\frac{c}{\sqrt{a^2+b^2+c^2}}z_1+\frac{d}{\sqrt{a^2+b^2+c^2}}$$

$$i.e \quad \pm\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}$$

Positive or negative sign is to be taken according as the expression is positive or negative in order to have distance always positive.

Example 1. A plane meets the co-ordinates axes in A , B , C such that the centroid of the triangle ABC is the point (p,q,r) , show that the equation of the plane is $\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3$.

Sol. Let the equation of the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ . ...(1)

It meet the axes in A ,B ,C.

$\therefore$ A (a,0,0), B(0,b,0) ,C (0,0,c) .

Centroid of the triangle ABC is

$$\frac{a+0+0}{3},\frac{0+b+0}{3},\frac{0+0+c}{3}$$

$$i.e.\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$$

But we are given at the Centroid is the point (p,q,r)

$\therefore \frac{a}{3}=p, \frac{b}{3}=q, \frac{c}{3}=r$ or a = 3p , b = 3q , c = 3r.

Putting the values of a, b , c in (1) , we get the required equation of the plane as

$\frac{x}{3p}+\frac{y}{3q}+\frac{z}{3r}=1$ or $\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3$

Example 2.A variable plane passes through a fixed point(p,q,r) and meets the co-ordinate axes in A,B,C . show that the locus of the point common to the planes through A,B.C parallel to the co-ordinate plane is $\frac{p}{x}+\frac{q}{y}+\frac{r}{z}=1$.

Sol. . Let the equation of the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. …………(1)

Since it passing through (p,q,r) ,

$\therefore \frac{p}{a}+\frac{q}{b}+\frac{r}{c}=1$ …………….(2)

The plane (1) meets the axes in A,B,C so that OA =a , OB = b, OC = c

Now equation of plane through A (a,0,0) and parallel to YZ plane is x = a

Similarly the equations of the planes through B (0,b,0) and C (0,0,c) and parallel to ZX and XY planes resp. are y = b and z = c

From equation (2) ,we get $\frac{p}{x}+\frac{q}{y}+\frac{r}{z}=1$.

Example 3. A variable plane moves so that the sum of reciprocal of its intercepts on the three co-ordinate axes is constant. Show that it passes through a fixed point.

Sol. Let the equation of the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. …………(1)

So that its intercepts on the axes are a,b,c.

Now we are given that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{k}(say)$$
$$\therefore \frac{k}{a}+\frac{k}{b}+\frac{k}{c}=1$$

Which shows that (1) passes through (K,k,k) which is a fixed point.

Example 4.(book 1105) Find the equation of plane which cuts equal intercept from the axes and passes through point (2,-1,5).

Sol. Let the equation of the plane be $\frac{x}{a}+\frac{y}{a}+\frac{z}{a}=1$.

i.e. x+y+z=a ………….(1)

it passes through point (2,-1,5)

$\therefore$ 2 -1 + 5 = a $\Rightarrow$ a=6

Put in (1) we get x +y +z =6.

Example 5. Prove that if a plane has a intercept a,b,c and is at a distance p units from the origin, then $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$.

Sol. The equation of the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. ………….(1)

Now , length of perpendicular from (0,0,0) to (1) is p.

$$\therefore \frac{|0+0+0-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=p$$
$$\Rightarrow \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}=\frac{1}{p}$$
$$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$$

Which is required condtion.

Example 6. A variable plane is at a constant distance p from origin and meets the axes in A, B and C resp. show that locus of the centroid of the triangle ABC is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{9}{p^2}$.

Sol. The equation of the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ . ...(1)

Now , length of perpendicular from (0,0,0) to (1) is p.

$$\therefore \frac{|0+0+0-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=p$$

$$\Rightarrow \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}=\frac{1}{p}$$

$$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2} \quad ....(2)$$

Centroid of the triangle ABC is

$$\frac{a+0+0}{3},\frac{0+b+0}{3},\frac{0+0+c}{3}$$

$$i.e.\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$$

Let Centroid of the triangle is the point (p,q,r)

$\therefore \frac{a}{3}=p,\frac{b}{3}=q,\frac{c}{3}=r$ or a = 3p , b = 3q , c = 3r.

Putting values of a,b,c in (2)

$$\frac{1}{9p^2}+\frac{1}{9q^2}+\frac{1}{9r^2}=\frac{1}{p^2}$$

$$i.e \frac{1}{p^2}+\frac{1}{q^2}+\frac{1}{r^2}=\frac{9}{p^2}$$

Hence, locus of centroid (p,q,r) is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{9}{p^2}$.

Example 7. Find the equation of two planes through the points (0,4,-3) , (6,-4,3) , which cuts off from the axes intercepts whose sum is zero.

Sol. Let the equation of the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ . ...(1)

We are given that a + b + c = 0 .... (2)

Equation (1) passes through the points (0 , 4,-3) and (6,-4,3)

$\therefore \frac{0}{a}+\frac{4}{b}-\frac{3}{c}=1$ ....(3)

$and \quad \frac{6}{a}-\frac{4}{b}+\frac{3}{c}=1$ ....(4)

adding (3) and (4) , we get

a =3

from (2) b + c = -3 $\Rightarrow$ b = -3 –c ....(5)

put in 3 , we have

$$\frac{4}{-3-c}-\frac{3}{c}=1$$

$$\Rightarrow \quad \frac{4}{3+c}+\frac{3}{c}+1=0$$

$$\Rightarrow \quad \frac{4c+9+3c}{c^2+3c}+1=0$$

$$\Rightarrow \quad \frac{4c+9+3c+c^2+3c}{c^2+3c}=0$$

$$\Rightarrow \quad c^2+10c+9=0$$

$\Rightarrow$ (c+1)(c+9) = 0 which gives c =-1 or -9

Put in (5) when c= -1, b = -2

when c= -9, b = 6

putting these values of a,b c in (1) we get

2x-3y-6z-6=0 and 6x +3y-2z-18=0

Example 8. The sum of the distances of any number of fixed points from a plane is zero. Show that the plane always passes through a fixed point.

Sol. Let equation of plane be ax + by + cz +d =0 .

Let the fixed points be $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots\ldots, (x_n, y_n, z_n)$.

$\because$ Sum of the distances of these points from the plane (1) =0

$$\frac{(ax_1+by_1+cz_1+d)}{\sqrt{a^2+b^2+c^2}}+\frac{(ax_2+by_2+cz_2+d)}{\sqrt{a^2+b^2+c^2}}+\ldots\ldots\ldots+\frac{(ax_n+by_n+cz_n+d)}{\sqrt{a^2+b^2+c^2}}=0$$

$$\Rightarrow a\left(\frac{\sum x_1}{\sqrt{a^2+b^2+c^2}}\right)+b\left(\frac{\sum y_1}{\sqrt{a^2+b^2+c^2}}\right)+c\left(\frac{\sum z_1}{\sqrt{a^2+b^2+c^2}}\right)+d\left(\frac{n}{\sqrt{a^2+b^2+c^2}}\right)=0 .$$

$$\Rightarrow a\left(\frac{\sum x_1}{n}\right)+b\left(\frac{\sum y_1}{n}\right)+c\left(\frac{\sum z_1}{n}\right)+d=0$$

Which shows that the plane (1) passing through the fixed point

$$\left(\frac{\sum x_1}{n},\frac{\sum y_1}{n},\frac{\sum z_1}{n}\right).$$

Example 9.Find the locus of a point the sum of the squares of whose distances from the planes x + y =0 , x-y=0 and x –z=0 is 5

Sol. Let $P(x_1, y_1, z_1)$ be any point .

Now sum sum of the squares of distances of P from the given planes =5.

$$\therefore \left(\frac{x_1+y_1}{\sqrt{1+1}}\right)^2+\left(\frac{x_1-y_1}{\sqrt{1+1}}\right)^2+\left(\frac{x_1-z_1}{\sqrt{1+1}}\right)^2=5$$

$$\Rightarrow \left(\frac{x_1^2+y_1^2+2x_1y_1}{2}\right)+\left(\frac{x_1^2+y_1^2-2x_1y_1}{2}\right)+\left(\frac{x_1^2+z_1^2-2x_1z_1}{2}\right)=5$$

$$\Rightarrow 3x_1^2+2y_1^2+z_1^2-2x_1z_1=10$$

Locus of P is $3x^2+2y^2+z^2-2xz-10=0$.

Example 10. Show that $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ is the incentre of the tetrahedron formed by the four planes x =0, y =0, z=0, 2x + 2y + z -1 =0.

Sol. The point P $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ is the incentre of tetrheadron formed by four planes if it is equidistance from the four given planes.

Now perpendicular distance of P from the plane x =0 is $\frac{\frac{1}{2}}{1}$.

perpendicular distance of P from the plane y =0 is $\frac{\frac{1}{2}}{1}$.

perpendicular distance of P from the plane x =0 is $\frac{\frac{1}{2}}{1}$

perpendicular distance of P from the plane x =0 is $\frac{1+1+\frac{1}{2}-1}{\sqrt{4+4+1}}=\frac{\frac{3}{2}}{3}=\frac{1}{2}$

since all the perpendicular distances are equal, $\therefore$ P is the incentre.

EXERCISE 1(b)

1. A plane meets the co-ordinates axes in A , B , C such that the centroid of the triangle ABC is the point (a,b,c) , show that the equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$.
2. A plane meets the co-ordinates axes in A , B , C such that the centroid of the triangle ABC is the point (1,-2,3) , show that the equation of the plane is 6x-3y+2z=18.
3. A variable plane is at a constant distance p from origin and meets the axes in A, B and C resp. show that locus of the centroid of the tetrahedron OABC is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p^2}$.
4. A variable plane is at a constant distance3p from origin and meets the axes in A, B and C resp. show that locus of the centroid of the triangle ABC is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$.
5. If the product of distance of point (1,1,1) from the origin and the plane x-y+z +k =0 be 5 , then find the value of k.

6.Prove that the locus of a point which moves so that the sum of its distances from any number of fixed planes is constant is a plane.

7. Prove that the locus of a point which moves so that the sum of its distances from two fixed planes fixed planes are in a constant ratio is a plane.

8.Find the locus of a point whose distance from the origin is 7 times the distance from the plane x +2y +2z -1=0.

ANSWERS

5. -6, 4

Article 4. Angle between two planes : By angle between two planes we mean angle between normals to the planes drawn from any point.

Let the two planes be $a_1x+b_1y+c_1z+d_1=0, a_2x+b_2y+c_2z+d_2=0$

The d.r's of the normals to the two planes are a_1, b_1, c_1 and a_2, b_2, c_2. Hence if θ be angle between them , then

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Cor 1.condition of perpendicularity.

If the two planes are perpendicular ,then their normal are also perpendicular.

The d.r's of the normals to the two planes are a_1, b_1, c_1 and a_2, b_2, c_2.

Thus the normals are perpendicular if $a_1a_2+b_1b_2+c_1c_2=0$.

Hence the planes are perpendicular if $a_1a_2+b_1b_2+c_1c_2=0$.

Cor 2.condition of parallelism.

If the two planes are parallel, then their normal are also parallel.

The d.r's of the normals to the two planes are a_1, b_1, c_1 and a_2, b_2, c_2.

$\therefore$ Condition is $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$.

Cor 3. Equation of any plane parallel to a given plane.

Let the given plane be ax + by +cz +d =0 …………….. (1)

and let any plane parallel to (1) be $a_1x+b_1y+c_1z+d_1=0$ ....(2)

since the planes are parallel, $\frac{a_1}{a}=\frac{b_1}{b}=\frac{c_1}{c}$

put in (2) we get $ax+by+cz+d_1=0$.

i.e. equation of two parallel planes differ by a constant.

Example 1. Find the angle between two planes 3x -6y +2z =7

And 2x + 2y - 2z =5.

Sol. The angle between two planes is the angle between their normals.

The d.r's of the normals to the two planes are <3,-6,2> and < 2,2,-2>

if θ be angle between them ,

$$\cos\theta=\frac{|(3)(2)+(-6)(2)+(2)(-2)|}{\sqrt{9+36+4}\sqrt{4+4+4}}=\frac{|-10|}{\sqrt{49}\sqrt{12}}=\frac{10}{7\times2\sqrt{3}}=\frac{5\sqrt{3}}{21}.$$

Hence $\theta=\cos^{-1}\left(\frac{5\sqrt{3}}{21}\right)$.

Example 2. Find the value of k so that the planes 2x +ky + 3z =15 and

x –y + 7kz = 13 are perpendicular.

Sol. If the two planes are perpendicular ,then their normal are also perpendicular

The d.r's of the normals to the two planes are <2,k,3> and < 1,-1,7k>

(2) (1) + (k) (-1) + (3) (7k) =0

$\Rightarrow$ 2 – k + 21k = 0

$\Rightarrow$20 k = -2

$\Rightarrow$ K = $\frac{-1}{10}$.

Example 3. Find the equation of a plane which passes through (1,-1,4) and is parallel to the plane 2x -3y +7z =11.

Sol. The equation of given plane is 2x -3y +7z =11.

The equation of plane parallel to the (1) is of form 2x – 3y + 7z + k = 0

$\because$ It passes through point (1,-1,4)

$\therefore$ 2 (1) – 3 (-1) + 7 (4) + k = 0

$\Rightarrow$ K + 33 = 0

$\Rightarrow$ K = -33

Hence equation of required plane is 2x -3y + 7z -33 =0

Example 4. Find the equation of the plane that bisects the segment line joinjng the points A(-1,2,3) ,B(3,6,-5) at right angle.

Sol. The mid point of the line joining the points A (-1,2,3) ,B(3,6,-5) is C(1,4,-1)

Equation of any plane through C(1,4,-1)

a (x – 1) + b (y – 4) + c (z + 1) = 0 ……….(1) (one point form)

The d.r's of line AB be < 3 + 1, 6 – 2, -5-3 > = < 4 , 4, -8>

Since the plane (1) is perpendicular to line AB

$\therefore$ Normal to a plane (1) is parallel to the line AB

$\therefore \frac{a}{4} = \frac{b}{4} = \frac{c}{-8}$

Putting these values of a,b,c in (1) we have

4(x-1) + 4 (y -4) – 8 (z + 1) = 0

Or 4x + 4y – 8z -28 =0

Example 5. Find equation of planes parallel to plane x + 2y -2z + 8 =0 which are at a distance of 2 units from the point (2,1,1).

Sol. Equation of any plane parallel to given plane is

x +2y -2z + K =0 ....(1)

Now distance of (2,1,1) from plane (2) = 2

$$\Rightarrow \frac{|2+2-2+k|}{\sqrt{1+4+4}} = 2 \Rightarrow \quad |k+2| = 6$$

$\Rightarrow$ K +2 = $\pm$ 6

$\Rightarrow$ K = 4 ,-8

EXERCISE 1(c)

1.Find the angle between following planes;

(a) 2x +2y -3z =5 and 3x-3y+5z=3

(b) 3x -4y +5z =0 and 2x –y -2z =0

(c) x+y-2z=3 and 2x-2y +z=2

2. Find the value of k for which the planes 3x – 6y -2z =7 and 2x +y –kz =5 are perpendicular to each other.

3. Find the equation of a plane which passes through (2,3,4) and is parallel to the plane 5x -6y +7z =3.

4. Find the equation of the plane that bisects the segment line joining the points A(1,2,3) ,B(3,4,5) at right angle.

5. Find equation of planes parallel to plane x -2y +2z -3 =0 which are at a distance of 1 units from the point (1,2,3).

ANSWERS

1. (a) $\cos^{-1}\left(\frac{15}{\sqrt{17\times 43}}\right)$ (b) $\frac{\pi}{2}$ (c) $\cos^{-1}\left(\frac{2}{3\sqrt{6}}\right)$.

2. 0

4.x+ y +z -9 =0

5.x-2y+2z=0; x-2y+2z-6=0

Article 5. Plane through the intersection of two given planes.

Find the equation of a plane passing through the intersection of two planes whose equations are $a_1x+b_1y+c_1z+d_1=0, a_2x+b_2y+c_2z+d_2=0$.

The equations of given planes are $a_1x+b_1y+c_1z+d_1=0, a_2x+b_2y+c_2z+d_2=0$.

Consider the equation $(ax_1+by_1+cz_1+d_1)+t(ax_2+by_2+cz_2+d_2)=0$ ....(1)

Where t is arbitrary constant.

Equation (1) is a linear equation in x,y,z

$\therefore$ Represents a plane.

Also equation (1) is satisfied by the co-ordinates of all those points which satisfy given equations.

Hence equation (1) represents a plane passing through the intersection of two planes.

Example 1. Find the equation of the plane through the intersection of the planes 3x – y +2z -4 =0 and x + y + z – 2 = 0 and the point (2,2,1).

Sol. The given planes are 3x – y +2z -4 =0 ....(1)

and x + y + z – 2 = 0 ....(2)

Any plane through the intersection of (1) and (2) is

$\therefore$ (3x – y +2z -4) + K (x + y + z – 2) = 0 ....(3)

Since it passes through (2,2,1)

$\Rightarrow$ (6 -2 + 2 -4) +k (2+ 2 +1 -2) =0

$\Rightarrow$ 2 + 3 k = 0 $\Rightarrow$ k = $\frac{-2}{3}$.

Putting in (3) , (3x – y +2z -4) $\frac{-2}{3}$ (x + y + z – 2) = 0

$\Rightarrow$ 9 x – 3y + 6z -12 – 2x -2y -2z + 4 = 0

$\Rightarrow$ 7x -5y + 4z -8 =0

Example 2. Find the equation of the plane through the line of intersection of the planes 2x +3y –z+1=0 and x + y -2 z +3 = 0 and perpendicular to the plane 3x – y -2z -4 =0.

Sol. The equation of the plane through the line of intersection of the planes 2x +3y –z+1=0 and x + y -2 z +3 = 0 is

(2x +3y –z+1) + k (x + y -2 z +3) =0 ....(1)

$\Rightarrow$ (2 +k) x + (3 + k) y – (1 + 2k) + 1 + 3k =0 ....(2)

Since plane (2) is perpendicular to plane 3x – y -2z -4 =0,

3 (2+k) + (-1) (3 + k) + (-2) (-1-2k) = 0

$\Rightarrow$ 6 + 3 k – 3 –k + 2 + 4k =0

$\Rightarrow$ 6k + 5 = 0 $\Rightarrow$ K = .

Putting in (1) , 12 x + 18 y – 6z + 6 – 5x -5 y + 10 z -15 =0

$\Rightarrow$ 7x + 13 y + 4z -9 =0

Example 3. Find the equation of the planes through the line of intersection of the planes x+3y +6=0 and 3x- y – 4z = 0 and which is at a unit distance from the origin.

Sol. The equation of the plane through the line of intersection of the planes x+3y +6=0 and 3x- y – 4z = 0 is

(x+3y +6) + k (3x- y – 4z) =0 ....(1)

(1+ 3k) x + (3 –k) y -4k + 6 =0 ....(2)

This is a unit distance from (0,0,0)

$$\therefore \frac{|6|}{\sqrt{(1+3k)^2+(3-k)^2+(-4k)^2}}=1$$

$$\Rightarrow 36=(1+3k)^2+(3-k)^2+(-4k)^2$$

$$\Rightarrow 36=1+9k^2+6k+9+k^2-6k+16k^2$$

$$\Rightarrow 36=26k^2+10$$

$$\Rightarrow 26=26k^2$$

$$\Rightarrow k^2=1 \Rightarrow k=\pm 1$$

Putting value of k in (1)

4x +2y-4z+6=0 and -2x + 4y +4z +6 =0

Example 4. . Find the equation of the plane through the intersection of the planes x+y+z-1=0 and 2x +3y –z +4 = 0 and parallel to x-axis.

Sol. The equation of the plane through the intersection of the planes x+y+z-1=0 and 2x +3y –z +4 = 0 is (x+y+z-1) + k (2x +3y –z +4) =0 ....(1)

$\Rightarrow$ (1 +2k) x+(1+3k)y + (1- k)z -1 +4k =0

Its d.r's are < 1+ 2k, 1+ 3k, 1-k >

Since plane is parallel to x-axis with d.r's < 1,0,0 >

(1) (1+ 2k) + 0 + 0 =0

$\Rightarrow$K = $\frac{-1}{2}$.

Putting in (2) , we get

$\Rightarrow$2x + 2y +2z -2 -2x-3y +z -4 =0

$\Rightarrow$ -y + 3z -6 =0

Example 5.The plane x – y –z -4 =0 is rotated through a right angle about its line of intersection with the plane x + y +2z =4. Find its equation in the new position.

Sol. Given planes are

x – y –z -4 =0 ....(1)

and x + y +2z -4=0 ....(2)

The equation of the plane through the intersection of the planes (1) and (2) is

(x – y –z -4) + k (x + y +2z -4) =0 ....(3)

$\Rightarrow$ (1+ k) x + (-1+k)y + (-1+2k) -4 – 4k =0 ....(4)

Now , plane (1) and (4) are perpendicular

(1) (1+k) + (-1) (-1+k) + (-1) (-1+2k) =0

$\Rightarrow$ 1+k +1-k +1-2k=0

$\Rightarrow$ 3- 2k =0 $\Rightarrow$ k = $\frac{3}{2}$.

Putting in (3)

2x -2y-2z-8 +3x+3y+6z-12=0

$\Rightarrow$ 5x +y +4z-20 =0

EXERCISE 1(d)

1 (a). Find the equation of the plane through the intersection of the planes x+y+z-6=0 and 2x+3y+4z+5= 0 and the point (1,1,1).

(b). Find the equation of the plane through the intersection of the planes 2x -3y +z-9=0 and x - y + z – 4 = 0 and the point (0,0,0).

2 (a). Find the equation of the plane through the line of intersection of the planes x +y –2z+3=0 and 3x - y -2 z -4 = 0 and perpendicular to the plane 2x +3y -z +1=0.

(b). Find the equation of the plane through the line of intersection of the planes 2x -y =0 and y -3 z = 0 and perpendicular to the plane 4x +5 y -3z -8 =0.

3. Find the equation of the planes through the line of intersection of the planes 2x+6y +12=0 and 3x- y – 4z = 0 and which is at a unit distance from the origin.

4 (a). Find the equation of the plane through the intersection of the planes 2x+y-z-3=0 and 5x -3y +4z +9= 0 and parallel to $\frac{x-1}{2}=\frac{y-3}{4}=\frac{z-5}{5}$.

(b). Find the equation of the plane through the intersection of the planes 3x-4y+5z-10=0 and 2x +2y -3z -4= 0 and parallel to x =2y =3z.

ANSWERS

1 (a) 20x +23y +26z-69=0 (b) x + 3y + 5z =0.

2 (a). 16x -12y -4z -43 =0 (b) 28x -17y +9z =0.

3 . 2x + y +2z +3 =0 , x -2y +2z -3 =0

4(a) . 7x + 9y -10z-27=0 (b) x-20y+27z-14=0

Article 6.Position of the origin w.r.t. the angle between two planes.

To prove that the quantity $a_1a_2+b_1b_2+c_1c_2$ is negative or positive according as the origin lies in the acute angle or obtuse angle between the planes $a_1x+b_1y+c_1z+d_1=0, a_2x+b_2y+c_2z+d_2=0$, d_1, d_2 being both positive.

The two given planes are $a_1x+b_1y+c_1z+d_1=0, a_2x+b_2y+c_2z+d_2=0$.

If the origin is in the acute angle between the planes ,then θ the angle between the normals from the origin to them is obtuse as in fig (1).

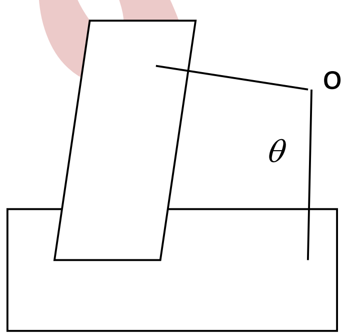
Cos θ is negative

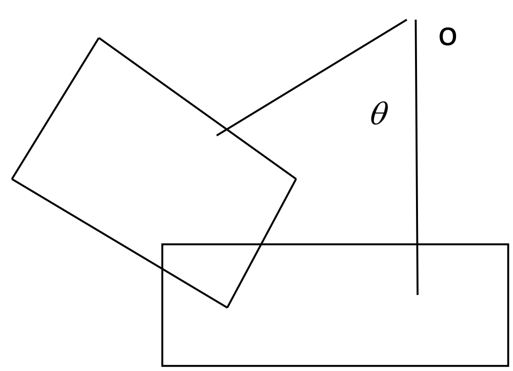
$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$ = -ve

$a_1a_2+b_1b_2+c_1c_2$ is negative

Similarly if the origin lies in obtuse angle between the planes then is acute in fig (2) and cos θ is +ve.

Or $a_1a_2+b_1b_2+c_1c_2$ is positive.





(1) (2)

Example 1. Is the origin in the acute or obtuse angle between the planes

2x – 3y + 4z + 2= 0 and x + 3y -2z – 5=0 ?

Sol. The equations of the two planes are 2x – 3y + 4z + 2= 0 and x + 3y -2z – 5=0

Making the constant terms +ve then equations of the two planes reduce to

2x – 3y + 4z + 2= 0 and -x - 3y +2z + 5=0.

Now $a_1a_2+b_1b_2+c_1c_2$ = (2) (-1) + (-3) (-3) +(4) (2) = -2 + 9 + 8 =15 = +ve

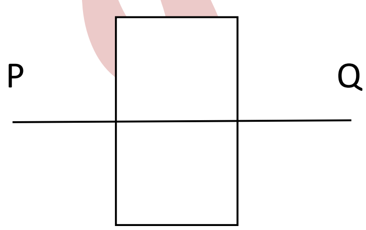
Hence origin lies in the acute angle between the planes.

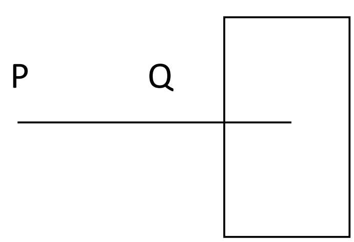
Article 7. Two sides of a plane.Show that two points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2)

Are on the same or opposite sides of the plane ax + by +cz +d =0 according as the expressions $ax_1+by_1+cz_1+d$ *and* $ax_2+by_2+cz_2+d$ are of the same or opposite signs.

Equation of the plane is ax + by +cz +d =0 ……………(1)

Let the line PQ be divided by the given plane (1) at R in the ratio k : 1. Then the co-ordinates of R is $\left(\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}, \frac{kz_2+z_1}{k+1}\right)$.





R lies on (1)

$$\therefore a\frac{kx_2+x_1}{k+1}+b\frac{ky_2+y_1}{k+1}+c\frac{kz_2+z_1}{k+1}+d=0$$
$$\Rightarrow a(kx_2+x_1)+b(ky_2+y_1)+c(kz_2+z_1)+d(k+1)=0$$
$$\Rightarrow k(ax_2+by_2+cz_2+d)+(ax_1+by_1+cz_1+d)=0$$
$$\Rightarrow k=\frac{-(ax_1+by_1+cz_1+d)}{(ax_2+by_2+cz_2+d)}$$

If $ax_1+by_1+cz_1+d$ and $ax_2+by_2+cz_2+d$ are of opposite signs , then k is +ve.

Plane (1) divides PQ internally in ratio K : 1 . Hence P and Q lie on the opposite sides of the plane .

If $ax_1+by_1+cz_1+d$ and $ax_2+by_2+cz_2+d$ are of same signs , then k is - ve.

Plane (1) divides PQ externally in ratio K : 1 . Hence P and Q lie on the same sides of the plane .

Example 2.Are the points (1,2,3) and (5 ,-2,-1) on the same or opposite sides of the plane 2x – 3y – z +2 =0 ?

Sol. The equation of given plane is 2x – 3y – z +2 =0.

For the point (1,2,3) , given plane is 2- 6 -3 +2 = -5 < 0

For the point (5 ,-2,-1), given plane is 10 +6 +1 +2 =19 > 0

Given points are on the opposite side of given plane .

Article 8. Equations of the planes bisecting the angle between the given planes.

Let the equations of the two planes be
$a_1x+b_1y+c_1z+d_1=0, a_2x+b_2y+c_2z+d_2=0$.

If (x,y,z) be the coordinates of any point on the plane bisecting the angles between the planes then the perpendicular distances of the point from two planes are equal numerically.

$$\therefore \ \frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

Simplifying we get the equations of the required planes bisecting the angle between the given planes.

Note ; How to distinguish between the two planes.

If we are to determine which of the two planes represents the plane bisecting the acute or obtuse angle between the planes, we should find the value of cos θ when θ is the angle between the bisecting plane and any of the given plane. From the value of cosθ obtained find the value of tanθ .

In case value of tanθ > 1 , that plane bisects the obtuse angle between them. In case value of tanθ < 1 , that plane bisects the acute angle between them.

Example 3.Find the equations of the planes bisecting the angle between the planes x + 2y +2z -9 =0 and 4x -3y +12z +13=0 and distinguish them.

Sol. The required equations of the planes bisecting the angle between the given planes are $\frac{x+2y+2z-9}{\sqrt{1+4+4}}=\pm\frac{4x-3y+12z+13}{\sqrt{16+9+144}}$.

25x +17y +62z -78 =0 and x +35 y -10z -156=0

Acute or obtuse.

Now angle between the plane x +2y+2z -9=0 and the bisecting plane x +35 y -10z -156=0 is given by

$$\cos\theta=\frac{(1)(1)+(2)(35)+(2)(-10)}{\sqrt{1+4+4}\sqrt{1+1225+100}}=\frac{51}{3\sqrt{1326}}=\frac{17}{\sqrt{1326}}$$

$$\therefore \tan\theta=\frac{\sqrt{1037}}{\sqrt{289}}>1$$

Hence the plane x +35 y -10z -156=0 bisects the obtuse angle between the planes and hence the other plane 25x +17y +62z -78 =0 bisects acute angle.

Example 4. Find the equations of the planes bisecting the angle between the planes 2x - y +2z +3 =0 and 3x -2y +6z +8=0 and distinguish them.

Sol. The required equations of the planes bisecting the angle between the given planes are

$$\frac{2x-y+2z+3}{\sqrt{4+1+4}}=\pm\frac{3x-2y+6z+8}{\sqrt{9+4+36}}$$
$$\Rightarrow 14x-7y+14z+21=\pm(9x-6y+18z+24)$$
$$\Rightarrow 5x-y-4z-3=0 \quad and \quad 23x-13y+32z+45=0$$

Acute or obtuse.

Now angle between the plane 2x - y +2z +3 =0 and the bisecting plane 5x-y-4z-3=0 is given by

$$\cos\theta=\frac{(2`)(5)+(-1)(-1)+(2)(-4)}{\sqrt{4+1+4}\sqrt{25+1+16}}=\frac{3}{3\sqrt{42}}=\frac{1}{\sqrt{42}}$$
$$\therefore \tan\theta=\sqrt{41}>1$$

Hence the plane 5x-y-4z-3=0 bisects the obtuse angle between the planes and hence the other plane 23x-13y+32z+45=0 bisects acute angle.

EXERCISE 1(e)

1.Is the origin in the acute or obtuse angle between the plane x + y-z-3=0 and x- 2y +z -3 =0 ?

2.Show that origin lies in the acute anle between the planes x + 2y +2z -9 =0 and 4x -3y +12z +13=0

3.Find the equation of the plane which bisects the acute angle between the planes 3x-4y+12z-26=0 and x +2y-2z-9=0.

4.Show that the plane 8x-14y-13=0 bisects the obtuse angle between the planes

4x+3y-5z=1=0 and 12x+5y-13z=0.

ANSWERS

1.acute angle

3.4x+38y-62z-39=0

Pair of planes

Article 9.To find the condition that the general homogeneous equation of second degee equation of second degree in x,y and z may represent a pair of planes and to find the angle between them and the condition that these two planes may be perpendicular to each other.

The general homogeneous equation of second degee equation of second degree in x,y and z is $ax^2+by^2+cz^2+2fyz+2gxz+2hxy=0$. ....(1)

Let the equation of the two planes represented by (1) be

$l_1x+m_1y+n_1z=0$ and $l_2x+m_2y+n_2z=0$

There will be no constant term in the two equations representing the planes, for otherwisetheir product will not be homogeneous

$$ax^2+by^2+cz^2+2fyz+2gxz+2hxy=(l_1x+m_1y+n_1z)(l_2x+m_2y+n_2z).$$

Comparing the coefficients , we have

$$l_1l_2=a, m_1m_2=b, n_1n_2=c,$$
$$m_1n_2+m_2n_1=2f, n_1l_2+n_2l_1=2g, l_1m_2+l_2m_1=2h. \qquad \text{................(2)}$$

The required condion is obtained by eliminating l_1, m_1, n_1 and l_2, m_2, n_2.

Now consider the product of two non zero determinants

$$\begin{vmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} \times \begin{vmatrix} l_2 & m_2 & n_2 \\ l_1 & m_1 & n_1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2l_1l_2 & l_1m_2+l_2m_1 & l_1n_2+l_2n_1 \\ l_2m_1+l_1m_2 & 2m_1m_2 & n_1m_2+n_2m_1 \\ l_1n_2+l_2n_1 & n_1m_2+n_2m_1 & 2n_1n_2 \end{vmatrix} = 0$$

On putting the value from (2) , We get

$$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

Is the required condition.

Again if θ be the angle between the two planes, then

$$\tan\theta = \frac{\sqrt{\sum(m_1n_2 - m_2n_1)^2}}{l_1l_2 + m_1m_2 + n_1n_2}$$

Now $\tan\theta = \dfrac{2\sqrt{f^2+g^2+h^2-ab-bc-ca}}{a+b+c}$

In case the planes are perpendicular then a + b + c =0

Example 1.Prove that the equation $2x^2 - 6y^2 - 12z^2 + 18yz + 2zx + xy = 0$ represents a pair of planes , find the angle between them.

Sol. The given equation can be written as a quadratic in x as

$$2x^2 + x(2z+y) - \left(6y^2 + 12z^2 - 18yz\right) = 0$$

$$\Rightarrow x = \frac{-(2z+y) \pm \sqrt{(2z+y)^2 + 4\times 2\times\left(6y^2 + 12z^2 - 18yz\right)}}{4}$$

$$\Rightarrow x = \frac{-(2z+y) \pm \sqrt{100z^2 + 49y^2 - 140yz}}{4}$$

$$\Rightarrow 4x = -(2z+y) \pm (10z - 7y)$$

X + 2y – 2z =0 ,2x -3y +6z =0.

Hence the given equation is factorized into two linear factors which represent two planes. If θ be the angle between the normals to the planes whose D.R's are <1,2,-2> , <2,-3,6>.

$$\cos\theta = \frac{(1)(2)+(2)(-3)+(-2)(6)}{\sqrt{1+4+4}\sqrt{4+9+36}} = \frac{16}{21}.$$

Example 2. Show that $\frac{a}{y-z} + \frac{b}{z-x} + \frac{c}{x-y} = 0$ represents a pair of planes.

Sol. The given equation can be written as

a (z-x) (x-y) + b (y-z) (x-y) + c (y-z) (z-x) =0.

$$\Rightarrow ax^2 + by^2 + cz^2 - (b+c-a)yz - (c+a-b)zx - (a+b-c)xy = 0$$

It will represents a pair of planes if

$$\begin{vmatrix} a & \frac{-(a+b-c)}{2} & \frac{-(c+a-b)}{2} \\ \frac{-(a+b-c)}{2} & b & \frac{-(b+c-a)}{2} \\ \frac{-(c+a-b)}{2} & \frac{-(b+c-a)}{2} & c \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$.

$$if \quad \begin{vmatrix} 0 & 0 & 0 \\ \frac{-(a+b-c)}{2} & b & \frac{-(b+c-a)}{2} \\ \frac{-(c+a-b)}{2} & \frac{-(b+c-a)}{2} & c \end{vmatrix} = 0$$

Which is true.

Example 3.Ifte equation f(x,y,z) = $ax^2+by^2+cz^2+2fyz+2gxz+2hxy=0$ represents a pair of planes ,prove that the product of the distances of the two planes from (α,β,γ) is $\frac{f(\alpha,\beta,\gamma)}{\sqrt{\sum a^2+4\sum f^2-2\sum bc}}$.

Sol. Let the equation of the two planes be

$l_1x+m_1y+n_1z=0$ and $l_2x+m_2y+n_2z=0$.

f(x,y,z) = $ax^2+by^2+cz^2+2fyz+2gxz+2hxy=(l_1x+m_1y+n_1z)(l_2x+m_2y+n_2z)$.

Comparing the coefficients , we have

$l_1l_2=a, m_1m_2=b, n_1n_2=c,$

$m_1n_2+m_2n_1=2f, n_1l_2+n_2l_1=2g, l_1m_2+l_2m_1=2h$.

If p_1 and p_2 be the perpendicular distance of the point (α,β,γ) from the two planes , then

$$p_1p_2=\frac{l_1\alpha+m_1\beta+n_1\gamma}{\sqrt{l_1^2+m_1^2+n_1^2}}\times\frac{l_2\alpha+m_2\beta+n_2\gamma}{\sqrt{l_2^2+m_2^2+n_2^2}}$$

$$=\frac{f(\alpha,\beta,\gamma)}{\sqrt{l_1^2l_2^2+m_1^2m_2^2+n_1^2n_2^2+\left(l_1^2m_2^2+m_1^2l_2^2\right)+\left(n_1^2l_2^2+l_1^2n_2^2\right)+\left(m_1^2n_2^2+n_1^2m_2^2\right)}}$$

$$=\frac{f(\alpha,\beta,\gamma)}{\sqrt{\sum l_1^2l_2^2+\sum(l_1m_2+l_2m_1)^2-2l_1m_2l_2m_1}}$$

$$=\frac{f(\alpha,\beta,\gamma)}{\sqrt{\sum a^2+4\sum f^2-2\sum bc}}.$$

EXERCISE 1(f)

1. Prove that the equation $2x^2-2y^2+4z^2+6zx+2yz+3xy=0$ represents a pair of planes , find the angle between them.
2. Prove that the equation $6x^2+4y^2-10z^2+4zx+3yz-11xy=0$ represents a pair of planes , find the angle between them.
3. Prove that the equation $2x^2+6y^2-12z^2+2zx+6yz+7xy=0$ represents a pair of planes , find the angle between them.

ANSWERS

1. $\theta=\cos^{-1}\left(\frac{4}{9}\right)$.
2. $\theta=90^\circ$.
3. $\theta=\cos^{-1}\left(\frac{4}{21}\right)$.

Article 10.If A_x, A_y and A_z be the projection of an area A on the co-ordinate planes , yz ,zx and xy planes recpectively , then $A^2 = A_x^2 + A_y^2 + A_z^2$.

A_x Is equal to the projection of area A on the yz plane.

$\therefore A_x$ =A cosα where α is the angle between the plane of area A and yz plane, i.e. between their normals .Let us suppose that l,m,n are the d.c's of the normal to the plane of area A. Then l=cosine of the angle between the normal and x-axis (Which is normal to the yz-plane) = cosα .

$\therefore A_x$ = Al. Similarly A_y = Am and A_z = An.

Squaring and adding , we get

$$A^2 = A_x^2 + A_y^2 + A_z^2$$

Cor 1. To find the area of the triangle , the co-ordinates of whose vertices are given

Now if (a,b,c) be the co-ordinates of a point P ,then the co-ordinates of the point P_x which is its projection on yz plane are (0,b,c) and P_y which is its projection on zx plane are (a,0,c) and P_z which is its projection on xy plane are (a,b,0)

Then we shall project the three given vertices on the yz plane and find the plane area A_x which will be projection of the triangle on the yz plane.

Similarly we shall find the values of A_y and A_z then $A^2 = A_x^2 + A_y^2 + A_z^2$.

Example 1.Find the area of a triangle the co-ordinates of whose vertices are (1,2,3) , (-2,1,-4) , (3,4,-2).

Sol. Co-ordinates of the points of projection in the yz plane of the three vertices are (0,2,3) ,(0,1,-4) and (0,4,-2) or in plane geometry these points are (2,3), (1,-4) and (4,-2).Then area

$$A_x = \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ 1 & -4 & 1 \\ 4 & -2 & 1 \end{vmatrix} = \frac{19}{2}$$

similarly $A_y = \frac{21}{2}, A_z = -2$.

$$A = \frac{\sqrt{1218}}{2}$$

Example 2.A plane makes intercept OA = a , OB = b , and OC = c respectively on the axis of co – ordinates. Find the area of the triangle ABC.

Sol. A_x = area of the plane triangle formed by the three points which are the projections of the vertices on the yz plane. Clearly the points B and C are in the yz plane and the projection of the point A (a,o,o) is (0,0,0) .

$\therefore A_x$ = area of triangle OBC ,which is right angle at O.

$A_x = \frac{1}{2}bc$ Similarly $A_y = \frac{1}{2}ca, A_z = \frac{1}{2}ab$.

$$A = \frac{1}{2}\sqrt{b^2c^2 + a^2c^2 + b^2a^2}$$

Example 3.From a point P (a,b,c) , a plane is drawn at right angles to OP to meet the co-ordinate axes at A,B,C. Prove that area of the triangle ABC is $\frac{r^5}{2abc}$ where OP= r.

Sol. The equation of the plane through P at right angle to OP is

a (x-a) + b (y-b) + c (x-c) =0

$$ax + by + cz = a^2 + b^2 + c^2 = r^2$$

The points A,B,C whee the plane meet the co-ordinate axes are

$$\left(\frac{r^2}{a},0,0\right),\left(0,\frac{r^2}{b},0\right),\left(0,0,\frac{r^2}{c}\right)$$

Now A_x = projection of triangle ABC on the yz plane

$$=\frac{1}{2}.OB.OC=\frac{1}{2}.\frac{r^2}{b}.\frac{r^2}{c}=\frac{r^4}{2bc}$$

$$similarly \quad A_y=\frac{1}{2}\frac{r^4}{ac}, A_z=\frac{r^4}{2ba}$$

$$\therefore A=\sqrt{A_x^2+A_y^2+A_z^2}$$

$$=\sqrt{\frac{r^8}{4b^2c^2}+\frac{r^8}{4a^2c^2}+\frac{r^8}{4b^2a^2}}.$$

$$=\frac{r^4}{2}\sqrt{\frac{1}{b^2c^2}+\frac{1}{a^2c^2}+\frac{1}{b^2a^2}}$$

$$=\frac{r^4}{2}\sqrt{\frac{a^2+b^2+c^2}{a^2b^2c^2}}=\frac{r^5}{2abc}$$

EXERCISE 1(g)

1. Find the area of a triangle the co-ordinates of whose vertices are (1,3,4) , (-2,5,3) , (2,4,-2).
2. Find the area of a triangle whivh is included between the plane 2x-3y +4z =12 and the coordinates planes.

Article 11. Volume of a tetrahedron

Find the volume of a tetrahedron in terms of the coordinates of its vertices.

Let $A(x_1,y_1,z_1), B(x_2,y_2,z_2), C(x_3,y_3,z_3)$ and $D(x_4,y_4,z_4)$ be the vertices of a tetrahedron ABCD . If V is the volume of a tetrahedron ,then

$$V=\frac{1}{2}\times\Delta ABC\times p$$

Where p =DM is the perpendicular from the vertex D to the opposite face ABC.

Further the equation of plane ABC is $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$. …………(1)

Let $\Delta_x, \Delta_y, \Delta_z$ be the orthogonal projection of Δ on the coordinates planes. Then we have

$$2\Delta_x = \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} = coefficien\,t\,of \quad x\,in \quad (1)$$

$$2\Delta_y = \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix} = coefficien\,t\,of \quad y\,in \quad (1)\,.$$

$$2\Delta_z = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = coefficien\,t\,of \quad z\,in \quad (1)$$

Also

$$p = \frac{1}{2\sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2}} \begin{vmatrix} x_4 & y_4 & z_4 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2\Delta} \begin{vmatrix} x_4 & y_4 & z_4 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}.$$

$$hence \quad v = \frac{1}{3} \times p \times \Delta = \frac{1}{6} \begin{vmatrix} x_4 & y_4 & z_4 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}$$

$$= \frac{1}{6} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

Example 1. Find the Volume of a tetrahedron whose vertices are (1,0,0) , (0,0,4) (0,0,2) (2,1,4).

Sol. Volume of a tetrahedron $= \frac{1}{6} \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 2 & 1 \\ 2 & 1 & 4 & 1 \end{vmatrix}$ Expand 1st coloumn

$$= \frac{1}{6}.1 \begin{vmatrix} 1 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \frac{1}{6} \times 2 = \frac{1}{3}$$

Example 2. A,B,C are the points (3,2,1) ,(-2,0,-3) and (0,0,-2) . Find the locus of P if the volume PABC = 5.

Sol . Let P be the point (x,y,z) .

volume PABC = 5

.

$$\pm\frac{1}{6}\begin{vmatrix} x & y & z & 1 \\ 3 & 2 & 1 & 1 \\ -2 & 0 & -3 & 1 \\ 0 & 0 & -2 & 1 \end{vmatrix} = 5$$

$$R_1 \rightarrow R_1 - R_4, R_2 \rightarrow R_2 - R_4, R_3 \rightarrow R_3 - R_4$$

$$\begin{vmatrix} x & y & z+2 & 0 \\ 3 & 2 & 3 & 0 \\ -2 & 0 & -1 & 0 \\ 0 & 0 & -2 & 1 \end{vmatrix} = \pm 30$$

exp *anding* C_4

$$\begin{vmatrix} x & y & z+2 \\ 3 & 2 & 3 \\ -2 & 0 & -1 \end{vmatrix} = \pm 30$$

X(-2) – y (3) + (z+2) 4 = ± 30

Or -2x -3y +4z +8 = ± 30.

Taking + ve sign , we have

-2x-3y+4z -22 =0

Taking –ve sign , we have

2x + 3y -4z -38 =0

Example 3. A, B ,C are three fixed points and a variable point P moves such that the volume of tetrahedron PABC is constant. Show that the locus of P is a plane Parallel to the ABC.

Sol. Let the fixed points be A (a,0,0) ,B (0,b,0, C (0,0,c) and let P(x,y,z) be the variable point. Now equation of plane ABC is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ....(1)

Again volume PABC = constant

$$\therefore \frac{1}{6}\begin{vmatrix} x & y & z & 1 \\ a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \end{vmatrix} = k$$

$$R_1 \to R_1 - R_4, R_2 \to R_2 - R_4, R_3 \to R_3 - R_4$$

$$\Rightarrow \begin{vmatrix} x & y & z-c & 0 \\ a & 0 & -c & 0 \\ 0 & b & -c & 0 \\ 0 & 0 & c & 1 \end{vmatrix} = 6k$$

Expanding R_4

$$\begin{vmatrix} x & y & z-c \\ a & 0 & -c \\ 0 & b & -c \end{vmatrix} = 6k$$

Or bcx +acy + abz –abc =6k

Or bcx +acy +abz =6 k +abc

Divided by abc $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\frac{6k}{abc}+1$

Which is a plane parallel to the plane (1)

Example 4. Prove that the four planes y +z =0 , z +x =0 , x + y =0 and x +y +z =p form a tetrahedron is $\frac{2}{3}p^3$.

Sol. The four planes of the tetrahedron are

y + z = 0 ....(1)

z + x =0 ....(2)

x + y =0 ....(3)

x + y + z = p ....(4)

From (1) , (2) and (3) we get x =0, y =0 , z=0

From (1) , (2) and (4) we get x =p, y =p , z= - p

From (1) , (3) and (4) we get x =p, y = -p , z= p

From (2) , (3) and (4) we get x = -p, y =p , z=p

Volume of the tetrahedron

$$=\frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ p & p & -p & 1 \\ p & -p & p & 1 \\ -p & p & p & 1 \end{vmatrix}$$

$$==\frac{1}{6}p^3\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{vmatrix}$$

exp *anding* R_1

$$=\frac{4}{6}p^3=\frac{2}{3}p^3$$

Example 5. A variable plane makes with the coordinate planes a tetrahedron of constant volume $64k^3$

Find the locus of the centroid of the tetrahedron.

Sol. Equations of the co-ordinate planes are x=0, y=0, z=0, ax + by + cz +d =0

The vertices of the tetrahedron formed by the four planes obtained by solving three of the four equations at a time are

O (0,0,0) , A ($\frac{-d}{a}$,0,0) , B (0, $\frac{-d}{b}$,0) and (0,0, $\frac{-d}{c}$)

Now volume of tetrahedron OABC = $64k^3$.

$$\pm\frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ \frac{-d}{a} & 0 & 0 & 1 \\ 0 & \frac{-d}{b} & 0 & 1 \\ 0 & 0 & \frac{-d}{c} & 1 \end{vmatrix} = 64k^3$$

$\exp anding \quad R_1$

$$\Rightarrow \begin{vmatrix} \frac{-d}{a} & 0 & 0 \\ 0 & \frac{-d}{b} & 0 \\ 0 & 0 & \frac{-d}{c} \end{vmatrix} = \pm 384k^3$$

$$\Rightarrow \frac{d}{a}\times\frac{d}{b}\times\frac{d}{c} = \pm 384k^3$$

$$\Rightarrow d^3 = \pm 384k^3 abc$$

The coordinates (α , β , γ) Of the centriod G of the tetrahedron OABC are given by

$$\alpha = \frac{0+(\frac{-d}{a})+0+0}{4} = \frac{-d}{4a}, \beta = \frac{-d}{4b}, \gamma = \frac{-d}{4c}$$

$$or \qquad a = \frac{-d}{4\alpha}, b = \frac{-d}{4\beta}, c = \frac{-d}{4\gamma}$$

Putting these values in (1) we get

$$d^3 = \pm 384k^3 \frac{d^3}{64\alpha\beta\gamma}$$

$$\alpha\beta\gamma = \pm 6k^3$$

Locus of (α , β , γ) is

$$xyz = \pm 6k^3$$

Exercise 1(h)

1. Find the volume of the tetrahedron whose vertices are (0,1,2) , (3,0,1) , (4,3,6) and (2,3,2)
2. A , B , C are the points (0,0,-1) , (-1,0,-2) , (1,2,3) . Find the locus of P if volume PABC =2.
3. Prove that the four planes my + nz =0, nz + lx =0, lx +my =0 and lx +my+nz =p form a tetrahedron whose volume is $\frac{2p^3}{3lmn}$.
4. A,B ,C,D are four coplanar points and A', B', C', D' their projection on any plane prove that volume of $AB'C'D'$ = - volume of $A'BCD$.
5. The lengths of the edges OA ,OB, OC of a tetrahedron OABC are a,b,c and angles BOC , COA , AOB are α, β, γ find the volume of the tetrahedron.
6. Show that the volume of the tetrahedron of which a pair of opppsite edges is formed by lengths r and s on the straight lines whose equations are $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}, \frac{x-a'}{l'} = \frac{y-b'}{m'} = \frac{z-c'}{n'}$ is $\frac{1}{6} rs \begin{vmatrix} a-a' & b-b' & c-c' \\ l & m & n \\ l' & m' & n' \end{vmatrix}$.

Answer

1. 6 (2) 2x + 3y – 2z -14 =0, 2x +3y -2z +10 =0

5. $\frac{1}{6} abc \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix}$